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Trial to Find a Simple Type Analytical Solution for the Equation of the Non-Fourier's Heat Conduction

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Abstract: *The theory of the Fourier's heat conduction has well predicted the temperature profiles of a system under non-equilibrium steady state. However, in cases of transient states and in nano-scale sizes, its application is seriously limited. Overcoming this limitation, the theory of the non-Fourier heat conduction was issued by C. Cattaneo and P. Vernotte. Although this new theory has been used in various thermal science areas, it requires a little considerable mathematical skill for obtaining an analytical solution. This study was performed to find a new and simple type solution for the hyperbolic type partial differential equation of the non-Fourier's heat conduction and, by inspecting each term included in the proposed solution, to verify a theoretical and a physical justification. The newly obtained analytical solution for the non-Fourier's heat conduction was a neat and simple type exponential function. It should be noticed that the purpose of this study does not propose the analytical solution for the non-Fourier's heat conduction but introduces the strategy to obtain the solution.*

Keywords: Heat propagation speed, non-Fourier's heat conduction, *Cattaneo-Vernotte* equation, heat wave

1. INTRODUCTION

The concept of the non-Fourier heat conduction was resulted from the newly issued heat conduction mechanism modifying the Fourier's heat conduction, which were issued by C. Cattaneo and P. Vernotte independently in 1958 [1]. The defect of the Fourier heat conduction equation was originally indicated by J. C. Maxwell in 1867 with the comments that its theory implies the infinite propagation speed of a thermal disturbance in a system [2].

The term of a wave implies that some physical quantities have the characteristics such as wave front, reflection, or refraction at a boundary. Additionally, if a heat transfer phenomenon can be treated by a wave theory, it should have the dual properties of wave and particle simultaneously since the heat transfer in solid can be interpreted by an imaginary particle called the phonon with a wavelike property [3,4].

On the other hand, some studies issued that the concept of a thermal wave had not validity because there are no observation of the wavelike characteristics and a thermal transient applied does not travel fast in a medium as a wave but diffuse in it [5,6].

The starting point of this study is to assume an appropriate solution for the partial differential equation of the non-Fourier's heat conduction as simple as possible, which is the only method to find the solution for a given differential equation. If this assumed solution satisfies the given partial differential equation, there is no reason that the assumed solution cannot be accepted as the solution of it. In this study, the exponential function with time and displacement as independent variables was assumed as a solution and it was shown that the assumed solution satisfies the differential equation of the Non-Fourier's heat conduction.

The physical conformity of an obtained analytical solution was not considered since this study is the first trial to find the solution with a simple form to meet the mathematical requirement. The exact solution to satisfy the mathematical and physical behaviors will be prepared by combining any discontinuity function into the obtained solution in later successive study.

The considered discontinuity functions are the step function and the Dirac-delta function and so on. However, it should be noticed that the purpose of this study does not propose the analytical solution for the non-Fourier's heat conduction but introduces the strategy to obtain the solution.

2. NON-FOURIER HEAT CONDUCTION

The partial differential equations for the non-Fourier heat conduction were presented by *C. Cattaneo* and *P. Vernotte*, and thereafter called the *Cattaneo-Vernotte* (C-V) equation. The C-V equation was resulted from modifying the Fourier's heat conduction equation of Eq. (1) with considering the time interval between a heat flux and a temperature gradient.

$$\frac{\partial^2 T(x,t)}{\partial t^2} = \alpha \frac{\partial^2 T(x,t)}{\partial x^2} . \tag{1}$$

The C-V equation is based on the finite propagation speed of a thermal disturbance at the system boundary or in a system, which means that the present heat flux should be determined by the temperature gradient at any earlier time, τ or vice versa. Therefore, the C-V equation has a little complicated comparing with Eq. (1).

$$\frac{\partial^2 T(x,t)}{\partial t^2} + \frac{1}{\tau}\frac{\partial T(x,t)}{\partial t} = \frac{\alpha}{\tau}\frac{\partial^2 T(x,t)}{\partial x^2} ; \tag{2}$$

$$\frac{\partial^2 q(x,t)}{\partial t^2} + \frac{1}{\tau}\frac{\partial q(x,t)}{\partial t} = \frac{\alpha}{\tau}\frac{\partial^2 q(x,t)}{\partial x^2} . \tag{3}$$

In the induction process of Eq. (2) and Eq. (3), only one dimensional system was considered for simplicity since the purpose of this study is to seek the feasibility of the simple type solution for the C-V equation.

In Eq. (1), Eq. (2) and Eq. (3), □□is the thermal diffusivity with the unit of m^2/s and □□is the thermal relaxation time, which means the time required to communicate among the neighboring constituents about any thermal disturbance occurred in their surroundings. Since the unit of □ is m^2/s^2. The square root of it has the unity of (m/s); therefore, it can be regarded as the propagation speed of a thermal wave mentioned in the previous section and will be symbolized as C_T. If ignoring the existence of the thermal relaxation time, Eq. (2) will turn back to the form of Eq. (1). This means that the C-V equation is more general than the conventional Fourier's heat conduction equation.

3. MATHEMATICAL SOLUTION OF THE C-V EQUATION

Since the temperature of a system is varied with the spatial and the time coordinates, the solution of Eq. (2) shall has the two independent variables of x and t, that is T=T(x,t). For relating x and t with the exponential function as a solution, the form of Eq. (4) is selected.

$$T(x,t) = A \cdot e^{ax} \cdot e^{bt} , \tag{4}$$

where *A*, *a* and *b* are the arbitrarily constants at present status.

From the inspection of the proposed solution, *a* and *b* of Eq. (4) shall have the unit of 1/m and 1/t respectively because the exponential function shall be only numeric value, which means that the constant *A* has to be the temperature although the exact values of those constants are not known at this stage.

After partially differentiating Eq. (4) with respect to x and t, Eq. (4) and substituting those partial derivatives into Eq. (2),

$$b^2 \cdot A \cdot e^{ax} \cdot e^{bt} + \frac{1}{\tau} b \cdot A \cdot e^{ax} \cdot e^{bt} = C_T^2 \cdot a^2 \cdot A \cdot e^{ax} \cdot e^{bt} . \tag{5}$$

In Eq. (5), C_T^2 is □/□ and its square root is the thermal wave propagation speed mentioned in the section 2. Cancelling the common factor in the both sides of Eq. (5),

$$b^2 + \frac{1}{\tau} b - C_T^2 \cdot a^2 = 0 . \tag{6}$$

From the inspection of each term of Eq. (6), it is confirmed that their units are identical to $1/s^2$ and maintain the dimensional homogeneity. In order that Eq. (4) has the qualification as a solution for the partial differential equation of Eq. (2), the condition of Eq. (6) shall be satisfied.

For finding the values of A, a and b of Eq. (4), the two boundary conditions are applied to it; $T(x,t)=T_h$ at t=0 and x=0; $T(x,t)=T_i$ at t=0 and x=l, where l is the length of a system as shown in Fig. 2.

From the two boundary conditions,

$$A = T_h ; \tag{7}$$

$$a = \frac{1}{l}\ln\left(\frac{T_i}{T_h}\right) . \tag{8}$$

From Eq. (7) and Eq. (8), it is found that A is just the temperature change imposed to a boundary and a has the unit of 1/m, therefore the requirements to the units of A and a are satisfied. Since the value of b is calculated from the quadratic formula of Eq. (6), the two values will be obtained.

$$b_1 = \frac{-\frac{1}{\tau} + \sqrt{\frac{1}{\tau^2} + 4C_T^2 \cdot a^2}}{2} ; \tag{9}$$

$$b_2 = \frac{-\frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} + 4C_T^2 \cdot a^2}}{2} . \tag{10}$$

From the inspection of Eq. (9) and Eq. (10), the unit of *b* is still 1/s, which is consistent with the requirement when selecting Eq. (4) as the solution for the C-V equation of Eq. (2).

Although the value of *b* is emerged as the two cases of Eq. (9) and Eq. (10), there would be no problem to find a general solution if applying the superposition principle to the obtained particular solutions [7]. For applying the superposition principle, the general solution of Eq. (2) will be

$$T(x,t) = A \cdot e^{ax} \cdot \left(e^{b_1 t} + e^{b_2 t}\right). \tag{11}$$

4. MATHEMATICAL VERIFICATION

The obtained or the assumed solution should be checked whether or not the original differential equation is satisfied with it. If applying the general solution of Eq. (11) to the C-V equation of eq. (2),

$$A \cdot e^{ax} \cdot \left(b_1^2 \cdot e^{b_1 t} + b_2^2 \cdot e^{b_2 t}\right) + \frac{1}{\tau} A \cdot e^{ax} \cdot \left(e^{b_1 t} + e^{b_2 t}\right) - \frac{\alpha}{\tau} a^2 \cdot A \cdot e^{ax} \cdot \left(e^{b_1 t} + e^{b_2 t}\right) = 0 \tag{12}$$

Cancelling the common term of Eq. (12),

$$\left(b_1^2 + \frac{b_1}{\tau} - \frac{\alpha}{\tau} a^2\right) \cdot e^{b_1 t} + \left(b_2^2 + \frac{b_2}{\tau} - \frac{\alpha}{\tau} a^2\right) \cdot e^{b_2 t} = 0 \tag{13}$$

If EXP(b_1t)=0 and EXP(b_2t)=0, the general solution of Eq. (11) will come out only the function of x. This means that the assumed general solution will be the function only dependent on the displacement, x, and therefore these conditions shall be excluded and the following two conditions be satisfied.

$$\left(b_1^2 + \frac{b_1}{\tau} - \frac{\alpha}{\tau} a^2\right) = 0\,; \tag{14}$$

$$\left(b_2^2 + \frac{b_2}{\tau} - \frac{\alpha}{\tau} a^2\right) = 0\,. \tag{15}$$

The confirmation, whether or not the conditions of Eq. (14) and Eq. (15) are satisfied, can be done by substituting the values of b_1 and b_2 into Eq. (14) and Eq. (15) respectively. Eq. (14) is mathematically verified by developing the following bracket expression and so does Eq. (15).

$$\left(b_1^2 + \frac{b_1}{\tau} - \frac{\alpha}{\tau} a^2\right) = \left(-\frac{1}{2\tau} + \frac{1}{2}\sqrt{\frac{1}{\tau^2} + 4\frac{\alpha}{\tau} a^2}\right)^2 + \frac{1}{\tau}\left(-\frac{1}{2\tau} + \frac{1}{2}\sqrt{\frac{1}{\tau^2} + 4\frac{\alpha}{\tau} a^2}\right) - \frac{\alpha}{\tau} a^2 = 0 \tag{16}$$

5. CONCLUSIONS

This study was started from the naive purpose to find a simple type general solution for the hyperbolic type partial differential equation of the non-Fourier's heat conduction, which is Eq. (2) called the *Cattaneo-Vernotte* equation. Up to now, many studies suggested the analytical solutions for the C-V equation. Even though taking it into consideration that they well predict the transient state of a system, they were very complicated shape, which was the barrier to the ones without mathematical ripe knowledge or skills.

The authors' concept for by-passing the mathematical complication is just to assume a simple type solution and to confirm whether or not the assumed solution satisfies the originally given differential equation. Through many tests of various functions, the simple exponential function of Eq. (4) was confirmed to satisfy the non-Fourier differential equation of Eq. (2), mathematically. However, to be a physically qualified, the obtained general solution shall be calculated and verified by using the experimental data.

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